Graph Theory

Some notions and definitions

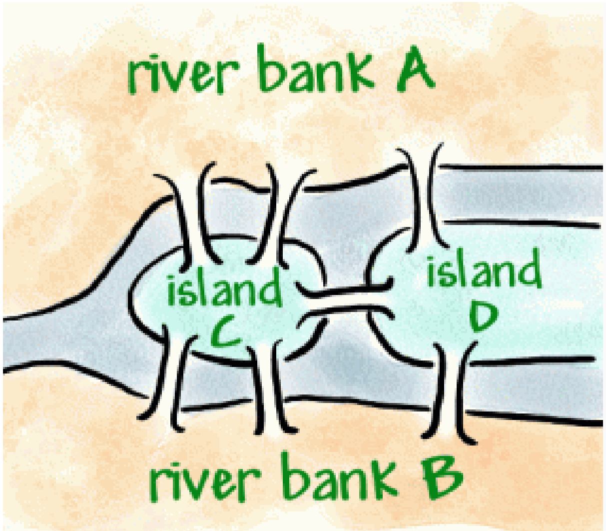
In mathematics, graph theory is the study of **graphs**, which are mathematical structures used to model pairwise relations between objects. A graph in this context is made up of **vertices** which are connected by **edges**.

The **degree** or **valency** of a vertex is the number of edges that meeting it, where an edge that connects a vertex to itself (a loop) is counted twice.

A **planar graph** is a graph that can be drawn on the plane in such a way that no edges cross each other.

Some history

Problem 1 (Königsberg bridges, 1736)

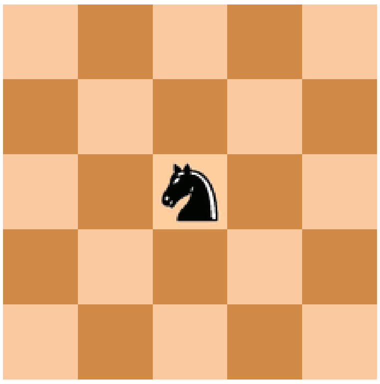


Is it possible to have a city walk passing through each bridge exactly once.

Hints:

One should first draw a graph on 4 vertices and 7 edges. It is clear that the vertices of odd valences should be either starting or ending points of the walk. So there might be at most 2 odd valences of vertices in case of positive answer. Therefore, the answer is **no**.

Problem 2 (knight's tours, in a paper from 1771)



Alexandre-Théophile Vandermonde
(1735 – 1796)

Is it possible for a knight to visit every square of the 5x5 blackboard only once?

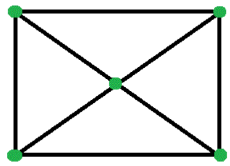
Hints:

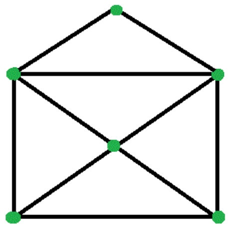
The answer is **yes**.

Valency of vertices

Problem 3

Is it possible to go through all edges for the following graphs (without making through any edge twice):

a) 

b) 

c) Any connected graph such that all vertices have even valencies except two of them.

Hints:

a) The answer is **yes**.
b) The answer is no. The reasoning is as in Problem 1.
c) The idea is: first to connect all the edges by several cycles and possible one non-cycling walk. Secondly re-glue two cycles passing through one vertex into one cycle (changing the direction at this vertex only). Now the problem follows from connectivity of the graph.

Problem 4 (Networks)

Please, help to design a new social network “Faceboo” which has 2n users such that:

1) **two** of them have exactly **1** friend each;
2) **two** of them have exactly **2** friends each;
…
n) the last **two** of them have exactly **n** friends each.

Hints:

The proof is by induction. There are several various constructions to do that. The one I was thinking about is as follows.

Induction hypothesis: Let for every k<n there exists a graphs on 2k vertices satisfying the condition of the formulation of a problem

Induction base: explicit constructions for n=2,3.

Induction step: Consider a graph satisfying all the conditions on 2n-4 vertices and add to points P and Q

1) Connect n-2 points with P.
2) Connect the remaining n-2 points with Q.
3) Connect P with Q.
4) Add new points R and S and connect them to P and Q respectively.

A direct calculation of valences shows that the step of induction is correct.

Planarity

<u>Euler's formula:</u> Assume a connected planar graph with V vertices and E edges splits the plane into F parts. Then

$$V - E + F = 2.$$

Problem 5 (Htrae Eht)

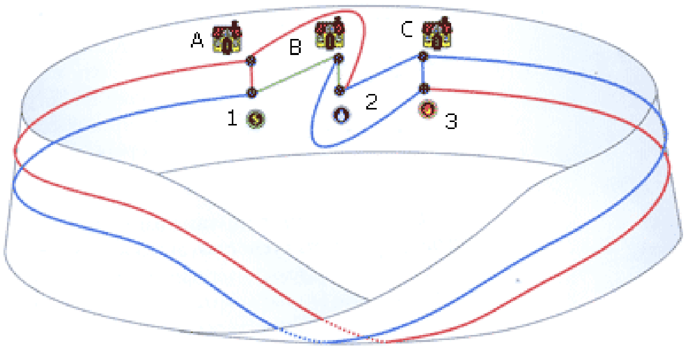
The planet Htrae Eht has 7 oceans connected by 10 non-crossing straits. It is also known that one can sail from any ocean to any other. How many continents does the Htrae Eht have?

Hints:

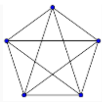
This is a direct corollary of the Euler's formula. Here V=7, E=10, and therefore F=2-7+10=5. We have 5 continents.

Problem 6 (gas-water-electricity)

Is it possible to connect three houses by tubes/wires to gas, water and electricity in such a way that the plan of the network does not have self-intersections?



Problem 7

Prove that the following graph is not planar: 

Hints:

The proof of both problems is based on two ideas.

1) Prove the following statement: Let ABCD be a square. Prove that it is impossible to join A and C; and B and D by passes such that
-- they do not intersect
-- they are both outside of the square

2) One can first draw a maximal cycle (on 6/5 vertices respectively) on the plane and reduce the problem to the case of ABCD in item 1) above.

Problem 8

Prove Euler's Formula.

Hints:

The proof is given by induction on the number of edges.

If the graph has a cycle, remove an edge from that cycle. Then V is the same while E and F are reduced by 1.

If the graph does not have a cycle then it is a tree and we can find one of the vertices of valence 1. When we remove it, F is the same while E and V decrease by 1.

Connectivity

Problem 9

Ten kids came to a birthday party. It turned out that every two of them have a grandpa in common. Prove that at least 7 of the 10 children the same grandpa.

Hints:

If they all have at most 3 grandpas, then there exists one with at least 7 grandchildren (as otherwise 6*3<10*2).
If they have more than 3 grandpa, then they all will have a common grandpa (this is not very hard to prove).

Problem 10

Every city in a country is connected to 100 roads. It is known that a traveller can reach every city from every other city. One of the roads was closed due to snowfalls. Prove that a traveller can still reach every city from every other city.

Hints:

Such network has a cycle passing through all the vertices (the proof is as in 3c), so removing an edge does not make this graph disconnected.

Problem 11*

A Union of Countries (UC) has 1001 cities. Each pair of the cities is connected by a one-way road. Each city has 500 outbound and 500 inbound roads. After a withdrawal of one of the countries from the UC, 668 cities remain in the UC. Prove that travellers can still travel from any city remaining in the UC to any other city remaining in the UC using the old system of roads and passing only cities remaining in the UC.

No hints: This is the hardest supplementary problem, so no hint is provided here.